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Commutation relation of Hecke operators for Arakawa lifting

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Abstract

The aim of this note is to make an announcement of our recent results [M-N] on Arakawa lifting, i.e. a theta lifting from elliptic cusp forms to automorphic forms on $Sp(1,q)$ (cf. [Ar-1], [N-1]). More precisely, restricting ourselves to the case of $q=1$, we reformulate Arakawa's lifting as a theta lifting from automorphic forms (f,f') on $GL_2\times B^{\times}$ to forms $\mathcal{L}(f,f')$ on $GSp(1,1)$, where $B^{\times}$ denotes the multiplicative group of a definite quaternion algebra over $\mathbb{Q}$. We show that this modified lifting satisfies a good commutation relation of Hecke operators. As an application we give all non-Archimedean local factors of spinor L-functions attached to the lifting in terms of Hecke eigenvalues for (f,f').

1

1.1 Notation

For an algebraic group $\mathcal{G}$ over $\mathbb{Q}$, $\mathcal{G}_v$ stands for the group of $\mathbb{Q}_v$-points of $\mathcal{G}$, where $\mathbb{Q}_v$ denotes the p-adic field (resp. the field of real numbers) when $v=p$ is a finite prime (resp. $v=\infty$). By $\mathcal{G}_{\mathbb{A}}$ (resp. $\mathcal{G}_{\mathbb{A},f}$), we denote the adelization of $\mathcal{G}$ (resp. the group of finite adeles in $\mathcal{G}_{\mathbb{A}}$). Let ψ be the additive character of $\mathbb{Q}_{\mathbb{A}}/\mathbb{Q}$ such that $\psi(x_{\infty})=\mathbf{e}(x_{\infty})$ for $x_{\infty}\in\mathbb{R}$, where we put $\mathbf{e}(z)=\exp(2\pi iz)$ for $z\in\mathbb{C}$. We denote by ψ_v the restriction of ψ to $\mathbb{Q}_v$ for a prime v of $\mathbb{Q}$.

1.2

Let B be a definite quaternion algebra over $\mathbb{Q}$. In what follows, we fix an identification between $B_{\infty}:=B\otimes_{\mathbb{Q}}\mathbb{R}$ and the Hamilton quaternion algebra $\mathbb{H}$, and an embedding $\mathbb{H}\hookrightarrow M_2(\mathbb{C})$. Let $B\ni b\mapsto\bar{b}\in B$ be the main involution of B, and put $\mathrm{tr}(b):=b+\bar{b}$ and $n(b):=b\bar{b}$ for $b\in B$. Let $B^{\times}:=B\backslash\{0\}$ be the multiplicative group of B. The center $Z(B^{\times})$ of $B^{\times}$ is $\mathbb{Q}^{\times}\cdot 1$. Let d_B be the discriminant of B. By definition, d_B is the product of finite primes p such that $B_p:=B\otimes_{\mathbb{Q}}\mathbb{Q}_p$ is a division algebra.

We let $G = GSp(1,1)$ be an algebraic group over $\mathbb{Q}$ defined by

$$G_{\mathbb{Q}} = \{g \in M_2(B) \mid {}^t\bar{g}Qg = \nu(g)Q,\ \nu(g) \in \mathbb{Q}^\times\},$$

where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Denote by Z_G the center of G.

The Lie group $G_\infty^1 := \{g \in G_\infty \mid \nu(g) = 1\}$ acts on the hyperbolic 4-space $\mathcal{X} := \{z \in \mathbb{H} \mid \mathrm{tr}(z) > 0\}$ by linear fractional transformations

$$g \cdot z := (az+b)(cz+d)^{-1}, \quad (g = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in G_\infty^1,\ z \in \mathcal{X}).$$

Let $\mu : G_\infty^1 \times \mathcal{X} \to \mathbb{H}^\times$ be the automorphy factor given by $\mu((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), z) := cz + d$. The stabilizer subgroup K_∞ of $z_0 := 1 \in \mathcal{X}$ in G_∞^1 is a maximal compact subgroup of G_∞^1, which is isomorphic to $Sp^*(1) \times Sp^*(1)$, where $Sp^*(1) := \{z \in \mathbb{H} \mid n(z) = 1\}$.

Let κ be a positive integer. Denote by $(\sigma_\kappa, V_\kappa)$ the representation of $\mathbb{H}$ given as

$$\mathbb{H} \hookrightarrow M_2(\mathbb{C}) \to \mathrm{End}(V_\kappa),$$

where the second arrow indicates the κ-th symmetric power representation of $M_2(\mathbb{C})$. Then

$$\tau_\kappa(k_\infty) := \sigma_\kappa(\mu(k_\infty, z_0)), \quad (k_\infty \in K_\infty)$$

gives rise to an irreducible representation of K_∞ of dimension $\kappa + 1$.

Define $\omega_\kappa : G_\infty^1 \to \mathrm{End}(V_\kappa)$ by

$$\omega_\kappa(g) := \sigma_\kappa(D(g))^{-1} n(D(g))^{-1}, \quad (g \in G_\infty^1),$$

where $D(g) := \frac{1}{2}(g \cdot z_0 + 1)\mu(g, z_0)$. It is known that ω_κ is a matrix coefficient of the discrete series representation with minimal K_∞-type (τ_κ, V_κ) (cf. [Ar-2, §2.6]). This discrete series is a quaternionic discrete series in the sense of B. Gross and N. Wallach [G-W]. We note that ω_κ is integrable if $\kappa > 4$.

Throughout the paper, we fix a maximal order $\mathcal{O}$ of B. We also fix a two-sided ideal $\mathfrak{A}$ of $\mathcal{O}$ satisfying the following conditions:

(i) If $p \not| d_B$, then $\mathfrak{A}_p = \mathcal{O}_p$.

(ii) If $p | d_B$, then $\mathfrak{A}_p = \mathfrak{P}_p^{e_p}$ with $e_p \in \{0, 1\}$, where $\mathfrak{P}_p$ is the maximal ideal of $\mathcal{O}_p$.

We set

$$D = \prod_{p | d_B, e_p = 0} p.$$

Note that $D = 1$ if and only if $e_p = 1$ for any $p | d_B$. Let $L := {}^t(\mathcal{O} \oplus \mathfrak{A}^{-1})$, which is a maximal lattice of $B^{\oplus 2}$. For a finite prime p, $K_p = \{k \in G_p \mid kL_p = L_p\}$ is a maximal compact subgroup of G_p, where $L_p := L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We set $K_f := \prod_{p<\infty} K_p$.

Definition 1.1. For an even integer $\kappa > 4$, let $\mathcal{S}_\kappa$ be the space of smooth functions $F : G_{\mathbb{A}} \to V_\kappa$ satisfying the following conditions:

1. $F(z\gamma g k_f k_\infty) = \tau_\kappa(k_\infty)^{-1}F(g) \quad \forall(z,\gamma,g,k_f,k_\infty) \in Z_{G,\mathbb{A}} \times G_{\mathbb{Q}} \times G_{\mathbb{A}} \times K_f \times K_\infty$,

2. F is bounded,

3. $c_\kappa \int_{G^1_\infty} \omega_\kappa(h_\infty^{-1} g_\infty) F(g_f h_\infty) dh_\infty = F(g_f g_\infty)$ for any fixed $(g_f, g_\infty) \in G_{\mathbb{A},f} \times G_\infty$,

where $c_\kappa := 2^{-4}\pi^{-2}\kappa(\kappa-1)$.

Here we remark that this automorphic form has been verified to be cuspidal (cf. [Ar-2, Proposition 3.1]) and to generate a quaternionic discrete series at the infinite place (cf. [N-2, Theorem 8.7]).

Next let H and H' be algebraic groups over $\mathbb{Q}$ defined by $H_{\mathbb{Q}} = GL_2(\mathbb{Q})$ and $H'_{\mathbb{Q}} = B^\times$ respectively, and denote by Z_H and $Z_{H'}$ the center of H and H' respectively. We define an action of $SL_2(\mathbb{R})$ on the complex upper half plane $\mathfrak{h} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ as usual. Let $U_\infty := \{h \in SL_2(\mathbb{R}) \mid h \cdot i = i\} = SO(2)$ and $U'_\infty := \{h' \in \mathbb{H} \mid n(h') = 1\} = Sp^*(1)$. Moreover, we put $U_f = \prod_{p<\infty} U_p$ and $U'_f = \prod_{p<\infty} U'_p$, where $U_p := \{u = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}_p) \mid c \in D\mathbb{Z}_p\}$ and $U'_p := \mathcal{O}_p^\times$.

Definition 1.2. (1) Let $S_\kappa(D)$ be the space of smooth functions f on $H_{\mathbb{A}}$ satisfying the following conditions:

1. $f(z\gamma h u_f u_\infty) = j(u_\infty, i)^{-\kappa} f(h) \quad \forall(z,\gamma,h,u_f,u_\infty) \in Z_{H,\mathbb{A}} \times H_{\mathbb{Q}} \times H_{\mathbb{A}} \times U_f \times U_\infty$,

2. For any fixed $h_f \in H_{\mathbb{A},f}$, $\mathfrak{h} \ni h_\infty \cdot i \mapsto j(h_\infty, i)^\kappa f(h_f h_\infty)$ is holomorphic for $h_\infty \in SL_2(\mathbb{R})$,

3. f is bounded,

where $j(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau) := c\tau + d$ denotes the standard $\mathbb{C}$-valued automorphy factor of $SL_2(\mathbb{R}) \times \mathfrak{h}$.
(2) Furthermore, $\mathcal{A}_\kappa$ stands for the space of smooth V_κ-valued functions f' on $H'_{\mathbb{A}}$ such that

$$f'(z'\gamma' h' u'_f u'_\infty) = \sigma_\kappa(u'_\infty)^{-1} f(h') \quad \forall(z',\gamma',h',u'_f,u'_\infty) \in Z_{H',\mathbb{A}} \times H'_{\mathbb{Q}} \times H'_{\mathbb{A}} \times U'_f \times U'_\infty.$$

2 Arakawa lift

2.1 Metaplectic representation

We fix a prime v of $\mathbb{Q}$. When $v = p$ is a finite prime (resp. $v = \infty$), $|*|_v$ denotes the p-adic valuation (resp. the usual absolute value for $\mathbb{R}$). For $X = \begin{pmatrix} x \\ y \end{pmatrix} \in B_v^{\oplus 2}$, we put $X^* := (\bar{x}, \bar{y})$. For a finite prime p, let $\mathbb{V}_p$ be the space of functions on $B_p^{\oplus 2} \times \mathbb{Q}_p^\times$ generated by $\varphi_1(X)\varphi_2(t)$, where φ_1 (resp. φ_2) is a locally constant and compactly supported function on $B_p^{\oplus 2}$ (resp. $\mathbb{Q}_p^\times$). We also let $\mathbb{V}_\infty$ be the space of smooth functions φ on $B_\infty^{\oplus 2} \times \mathbb{Q}_\infty^\times = \mathbb{H}^{\oplus 2} \times \mathbb{R}^\times$ such that, for any fixed $t \in \mathbb{R}^\times$, $X \mapsto \varphi(X,t)$ is rapidly decreasing on $\mathbb{H}^{\oplus 2}$.

Lemma 2.1. *There exists a smooth representation $r = r_v$ of $G_v \times H_v \times H_v'$ on $\mathbb{V}_v$ given as follows:*
For $\varphi \in \mathbb{V}_v$, $X \in B_v^{\oplus 2}$ and $t \in \mathbb{Q}_v^\times$,

$$r(g,1,1)\varphi(X,t) = |\nu(g)|_v^{-\frac{3}{2}}\varphi(g^{-1}X, \nu(g)t), \quad (g \in G_v), \tag{2.1}$$

$$r(1, (\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}), 1)\varphi(X,t) = \psi_v(\frac{bt}{2}\operatorname{tr}(X^*QX))\varphi(X,t), \quad (b \in \mathbb{Q}_v), \tag{2.2}$$

$$r(1, (\begin{smallmatrix} a & 0 \\ 0 & a' \end{smallmatrix}), 1)\varphi(X,t) = |a|_v^{\frac{7}{2}}|a'|_v^{-\frac{1}{2}}\varphi(aX, (aa')^{-1}t), \quad (a, a' \in \mathbb{Q}_v^\times), \tag{2.3}$$

$$r(1, (\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}), 1)\varphi(X,t) = |t|_v^4 \int_{B_v^{\oplus 2}} \psi_v(t\operatorname{tr}(Y^*QX))\varphi(Y,t)d_QY, \tag{2.4}$$

$$r(1,1,z)\varphi(X,t) = |n(z)|_v^{\frac{3}{2}}\varphi(Xz, n(z)^{-1}t), \quad (z \in B_v^\times). \tag{2.5}$$

Here d_QY is the Haar measure on $B_v^{\oplus 2}$ self-dual with respect to the pairing

$$B_v^{\oplus 2} \times B_v^{\oplus 2} \ni (Y, Y') \mapsto \psi_v(\operatorname{tr}(Y^*QY')).$$

2.2

When $v = p < \infty$, we put

$$\varphi_{0,p}(X,t) := \operatorname{char}_{L_p}(X)\operatorname{char}_{\mathbb{Z}_p^\times}(t),$$

where $\operatorname{char}_{L_p}$ (resp. $\operatorname{char}_{\mathbb{Z}_p^\times}$) is the characteristic function of $L_p = {}^t\left(\mathcal{O}_p \oplus \mathfrak{A}_p^{-1}\right)$ (resp. $\mathbb{Z}_p^\times$). When $v = \infty$, we put

$$\varphi_{0,\infty}^\kappa(X,t) := \begin{cases} t^{\frac{\kappa+3}{2}}\sigma_\kappa((1,1)X)\mathbf{e}(\frac{it}{2}\operatorname{tr}(X^*X)) & (t > 0) \\ 0 & (t < 0) \end{cases}.$$

Let $\mathbb{V}_\mathbf{A}$ be the restricted tensor product of $\mathbb{V}_v$ with respect to $\{\varphi_{0,p}\}_{p<\infty}$. By $r_\mathbf{A}$ we denote a smooth representation of $G_\mathbf{A} \times H_\mathbf{A} \times H_\mathbf{A}'$ on $\mathbb{V}_\mathbf{A}$ given as

$$r_\mathbf{A}(g,h,h')\varphi := \bigotimes_v r_v(g_v, h_v, h_v')\varphi_v$$

for $\varphi = \bigotimes_v \varphi_v \in V_\mathbf{A}$ and $(g = (g_v), h = (h_v), h' = (h_v')) \in G_\mathbf{A} \times H_\mathbf{A} \times H_\mathbf{A}'$.

We define a function $\varphi_0^\kappa \in \mathbb{V}_\mathbf{A}$ by

$$\varphi_0^\kappa(X,t) := \varphi_{0,\infty}^\kappa(X_\infty, t_\infty)\prod_{p<\infty}\varphi_{0,p}(X_p, t_p)$$

for $X = (X_v) \in B_{\mathbb{A}}^{\oplus 2}$ and $t = (t_v) \in \mathbb{Q}_{\mathbb{A}}^{\times}$, and set

$$\theta^{\kappa}(g,h,h') := \sum_{(X,t)\in B^{\oplus 2}\times\mathbb{Q}^{\times}} r_{\mathbb{A}}(g,h,h')\varphi_0^{\kappa}(X,t), \quad ((g,h,h') \in G_{\mathbb{A}} \times H_{\mathbb{A}} \times H'_{\mathbb{A}}). \tag{2.6}$$

This series is uniformly convergent on any compact subset of $G_{\mathbb{A}} \times H_{\mathbb{A}} \times H'_{\mathbb{A}}$, and satisfies

$$\theta^{\kappa}(\gamma g k_f k_{\infty}, \gamma_1 h u_f u_{\infty}, \gamma_2 h' u'_f u'_{\infty}) = \tau_{\kappa}(k_{\infty})^{-1} j(u_{\infty}, i)^{-\kappa}\theta(g,h,h')\sigma_{\kappa}(u'_{\infty})$$

for $(\gamma, g, k_f, k_{\infty}) \in G_{\mathbb{Q}} \times G_{\mathbb{A}} \times K_f \times K_{\infty}$, $(\gamma_1, h, u_f, u_{\infty}) \in H_{\mathbb{Q}} \times H_{\mathbb{A}} \times U_f \times U_{\infty}$ and $(\gamma_2, h', u'_f, u'_{\infty}) \in H'_{\mathbb{Q}} \times H'_{\mathbb{A}} \times U'_f \times U'_{\infty}$. It is also verify that θ^{κ} is $Z_{G,\mathbb{A}} \times Z_{H,\mathbb{A}} \times Z_{H',\mathbb{A}}$-invariant.

For $f \in S_{\kappa}(D)$ and $f' \in \mathcal{A}_{\kappa}$, we set

$$\mathcal{L}(f,f')(g) := \int_{Z_{H,\mathbb{A}}H_{\mathbb{Q}}\backslash H_{\mathbb{A}}} dh \int_{Z_{H',\mathbb{A}}H'_{\mathbb{Q}}\backslash H'_{\mathbb{A}}} dh'\, \theta^{\kappa}(g,h,h')\overline{f(h)}f'(h') \quad (g \in G_{\mathbb{A}}). \tag{2.7}$$

Theorem 2.2 (Arakawa, Narita). *Suppose* $\kappa > 6$.
(i) The integral (2.7) *is absolutely convergent.*
(ii) $\mathcal{L}(f,f')(g) \in S_{\kappa}$.

Proof. Since $G_{\mathbb{A}} = Z_{G,\mathbb{A}}G_{\mathbb{Q}}G_{\infty}^1 K_f$ (cf. [Shim-2, Theorem 6.14]), it is sufficient to consider the restriction of $\mathcal{L}(f,f')$ to G_{∞}^1. By a standard argument, we see that $\mathcal{L}(f,f')|_{G_{\infty}^1}$ is a finite linear combination of original Arakawa lift (cf. [Ar-1], [N-1, §4] and [N-3, Theorem 4.1]), from which the theorem follows. □

Remark 2.3. At the Archimedean place our lifting reads a correspondence between the quaternionic discrete series of G_{∞}^1 with minimal K_{∞}-type τ_{κ} and the discrete series representation of $O^*(4) \simeq SL_2(\mathbb{R}) \times SU(2)$ given by the direct product of the holomorphic disrete series of $SL_2(\mathbb{R})$ with Blattner parameter κ and the representation σ_{κ} of $Sp^*(1) \simeq SU(2)$. This is compatible with the result [J] of J. S. Li on theta correspondences for unitary representations with non-zero cohomology (cf. [J, §6, (I_1)]).

For the case of $GSp(1,q)$ we would be able to give an adelic reformulation of the lifting similarly. In view of [N-3, Theorem 4.1] and [J, §6, (I_1)], the weight of elliptic cusp forms or the Blatter parameter of the holomorphic discrere series of $SL_2(\mathbb{R})$ should be $\kappa - 2q + 2$ for this general case.

3 Main results

3.1

To state our results, we need to review several facts on Hecke operators.

3.2

First we consider the case where $p \nmid d_B$. We fix an isomorphism of B_p onto $M_2(\mathbb{Q}_p)$ such that $\mathcal{O}_p$ maps onto $M_2(\mathbb{Z}_p)$ and that the main involution of B_p corresponds to an involution of $M_2(\mathbb{Q}_p)$ given by

$$M_2(\mathbb{Q}_p) \ni X \mapsto w^{-1}\,{}^t\!Xw, \quad (w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)).$$

The reduced trace tr corresponds to the trace Tr of $M_2(\mathbb{Q}_p)$. We henceforth identify B_p with $M_2(\mathbb{Q}_p)$ using the above isomorphism. Then G_p, K_p, H'_p and U'_p are identified with $GSp(J, \mathbb{Q}_p)$, $GSp(J, \mathbb{Z}_p)$, $GL_2(\mathbb{Q}_p)$ and $GL_2(\mathbb{Z}_p)$ respectively, where $GSp(J)$ is the group of similitudes of $J = \left(\begin{smallmatrix} 0 & w \\ w & 0 \end{smallmatrix}\right)$. Note that we can identify U_p with U'_p by the isomorphism $B_p \simeq M_2(\mathbb{Q}_p)$ fixed above.

Define Hecke operators $\mathcal{T}_p^i$ $(i = 0, 1, 2)$ on $\mathcal{S}_\kappa$ by

$$\mathcal{T}_p^i F(g) = \int_{G_p} F(gx)\Phi_p^i(x)dx,$$

where Φ_p^0, Φ_p^1 and Φ_p^2 are the characteristic function of $K_p \operatorname{diag}(p, p, p, p)K_p$, $K_p \operatorname{diag}(p, p, 1, 1)K_p$ and $K_p \operatorname{diag}(p^2, p, p, 1)K_p$ respectively. Note that $\mathcal{T}_p^0 F = F$ for any $F \in \mathcal{S}_\kappa$.

We also define Hecke operators T_p and T'_p on $S_\kappa(D)$ and $\mathcal{A}_\kappa$ by

$$T_p f(h) = \int_{H_p} f(hx)\phi_p(x)dx,$$

$$T'_p f'(h') = \int_{H'_p} f'(h'x')\phi'_p(x')dx',$$

where $\phi_p = \phi'_p$ is the characteristic function of $GL_2(\mathbb{Z}_p) \operatorname{diag}(p, 1) GL_2(\mathbb{Z}_p)$.

3.3

We next consider the case where $p | d_B$, i.e., B_p is a division algebra. In this case, we fix a prime element Π of B_p and put $\pi := n(\Pi)$. Then π is a prime element of $\mathbb{Q}_p$.

Define Hecke operators $\mathcal{T}_p^i$ $(i = 0, 1)$ on $\mathcal{S}_\kappa$ by

$$\mathcal{T}_p^i F(g) = \int_{G_p} F(gx)\Phi_p^i(x)dx,$$

where Φ_p^0 and Φ_p^1 are the characteristic functions of $K_p \cdot \operatorname{diag}(\Pi, \Pi) \cdot K_p$ and $K_p \operatorname{diag}(1, \pi)K_p$ respectively. Note that $(\mathcal{T}_p^0)^2 F = F$ for any $F \in \mathcal{S}_\kappa$. We also define Hecke operators T_p and T'_p on $S_\kappa(D)$ and $\mathcal{A}_\kappa$ by

$$T_p f(h) = \int_{H_p} f(hx)\phi_p(x)dx,$$

$$T'_p f'(h') = \int_{H'_p} f'(h'x')\phi'_p(x')dx'.$$

Here ϕ'_p is the characteristic function of $U'_p\Pi U'_p = \Pi U'_p$ and ϕ_p is defined as follows: If $p|D$, ϕ_p is the sum of the characteristic functions of $U_p(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix})U_p$ and $U_p(\begin{smallmatrix}1 & 0\\ 0 & \pi\end{smallmatrix})U_p$. If $p \nmid D$, ϕ_p is the characteristic function of $U_p(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix})U_p$.

3.4

We say that $F \in \mathcal{S}_\kappa$ is a *Hecke eigenform* if F is a common eigenfunction of the Hecke operators $\mathcal{T}_p^i$ for any $p < \infty$. Let $F \in \mathcal{S}_\kappa$ be a Hecke eigenform with $\mathcal{T}_p^i F = \Lambda_p^i F$ ($\Lambda_p^i \in \mathbb{C}$). We define the spinor L-function of F by

$$L(F, spin, s) = \prod_{p<\infty} L_p(F, spin, s),$$

where $L_p(F, spin, s) = Q_p(F, p^{-s})^{-1}$,

$$Q_p(F,t) = \begin{cases} 1 - p^{\kappa-3}\Lambda_p^1 t + p^{2\kappa-5}(\Lambda_p^2 + p^2 + 1)t^2 - p^{3\kappa-6}\Lambda_p^1 t^3 + p^{4\kappa-6}t^4 & \text{if } p \not| d_B, \\ 1 - \{p^{\kappa-3}\Lambda_p^1 - p^{\kappa-3}(p^{A_p} - 1)\Lambda_p^0\}t + p^{2\kappa-3}(\Lambda_p^0)^2 t^2 & \text{if } p | d_B, \end{cases}$$

and

$$A_p = \begin{cases} 1 & \text{if } p \not| D, \\ 2 & \text{if } p|D. \end{cases}$$

The Euler factor for $p \nmid d_B$ (resp. $p|d_B$) is given by the formula for the denominator of the Hecke series in [Shim-1, Theorem 2] (resp. [H-S, §4] and [Su, (1-34)]), under the normalization of the Hecke eigenvalues

$$\begin{cases} (\Lambda_p^0, \Lambda_p^1, \Lambda_p^2) \to (p^{2(\kappa-3)}\Lambda_p^0, p^{\kappa-3}\Lambda_p^1, p^{2(\kappa-3)}\Lambda_p^2) & (p \nmid d_B) \\ (\Lambda_p^0, \Lambda_p^1) \to (p^{\kappa-3}\Lambda_p^0, p^{\kappa-3}\Lambda_p^1) & (p|d_B) \end{cases}.$$

We say that $f \in S_\kappa(D)$ (resp. $f' \in \mathcal{A}_\kappa$) is a *Hecke eigenform* if f (resp. f') is a common eigenfunction of T_p (resp. T'_p) for any $p < \infty$. For Hecke eigenforms $f \in S_\kappa(D)$ and $f' \in \mathcal{A}_\kappa$ with $T_p f = \lambda_p f$ and $T'_p f' = \lambda'_p f'$ (λ_p, $\lambda'_p \in \mathbb{C}$), we define L-functions

$$L^D(f,s) = \prod_{p\nmid D} \left(1 - \lambda_p p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1},$$
$$L^{d_B}(f',s) = \prod_{p\nmid d_B} \left(1 - \lambda'_p p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1}.$$

When $D = 1$, we write $L(f,s)$ for $L^D(f,s)$, which is the usual Hecke L-function of f.

3.5

We are now able to state the main result.

Theorem 3.1. *Let $f \in S_\kappa(D)$ and $f' \in \mathcal{A}_\kappa$, and suppose that*

$$T_p f = \lambda_p f, \quad T'_p f' = \lambda'_p f'$$

for each $p < \infty$. Then $F(g) := \mathcal{L}(f, f')(g)$ is a Hecke eigenform and the Hecke eigenvalues are given as follows:

(i) If $p \nmid d_B$, we have

$$\mathcal{T}_p^0 F = F,$$
$$\mathcal{T}_p^1 F = \left(p\overline{\lambda_p} + p\lambda'_p\right) F,$$
$$\mathcal{T}_p^2 F = \left(p\overline{\lambda_p}\lambda'_p + p^2 - 1\right) F.$$

(ii) If $p | d_B$, we have

$$\mathcal{T}_p^0 F = \lambda'_p F,$$
$$\mathcal{T}_p^1 F = \left(p\overline{\lambda_p} + (p-1)\lambda'_p\right) F.$$

Remark 3.2. Noting that the elliptic cusp forms are assumed to have the trivial central character, we see that all the Hecke operators above for the cusp forms are self-adjoint with respect to the Petersson inner product. We can thus remove the complex conjugates of their Hecke eigenvalues in the formula above.

Corollary 3.3. *Let f and f' be as in Theorem 3.1. Then we have*

$$L(\mathcal{L}(f, f'), spin, s) = L^D(f, s)\, L^{d_B}(f', s) \prod_{p|D} \left(1 - \{\lambda_p + (1-p)\lambda'_p\} p^{\kappa-2-s} + p^{2\kappa-3-2s}\right)^{-1}.$$

In particular, if $D = 1$, we have

$$L(\mathcal{L}(f, f'), spin, s) = L(f, s)\, L^{d_B}(f', s).$$

The results above are deduced from the commutation relation of Hecke operators for the metaplectic representation r as follows:

Proposition 3.4. *For a function ϕ on H_p, we put $\widehat{\phi}(h) = \phi(h^{-1})$ $(h \in H_p)$. We define $\widehat{\phi}'$ for $\phi' \colon H'_p \to \mathbb{C}$ in a similar manner.*
(1) *Suppose that $p \nmid d_B$. Then we have*

(*i*) $r(\Phi_p^1, 1, 1)\varphi_{0,p} = p \cdot r(1, \widehat{\phi}_p, 1)\varphi_{0,p} + p \cdot r(1, 1, \widehat{\phi}'_p)\varphi_{0,p}$,
(*ii*) $r(\Phi_p^2, 1, 1)\varphi_{0,p} + (1 - p^2)r(\Phi_p^0, 1, 1)\varphi_{0,p} = p \cdot r(1, \widehat{\phi}_p, \widehat{\phi}'_p)\varphi_{0,p}$.
(2) *Suppose that* $p|d_B$. *Then we have*

$$r(\Phi_p^0, 1, 1)\varphi_{0,p} = r(1, 1, \widehat{\phi}'_p)\varphi_{0,p},$$
$$r(\Phi_p^1, 1, 1)\varphi_{0,p} = p \cdot r(1, \widehat{\phi}_p, 1)\varphi_{0,p} + (p - 1)r(1, 1, \widehat{\phi}'_p)\varphi_{0,p}.$$

Remark 3.5. When $p \nmid d_B$ the formula for the Hecke eigenvalues is essentially the same as the corresponding result of Yoshida lifting (cf. [Y, Theorem 6.1]). For such p this leads to the following decomposition

$$L_p(\mathcal{L}(f, f'), spin, s) = (1 - \lambda_p p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}(1 - \lambda'_p p^{\kappa-2-s} + p^{2\kappa-3-2s})^{-1}.$$

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